Lezione 24

(M, ∇) varietà con connessione $\dashrightarrow T$ torsione

Abbiamo visto tre tipi di funzioni bilineari su $\mathbb{R}$ $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

- $[X,Y] \longleftarrow [X,Y]$ dipende da X e Y in $U(p)$
- $\nabla_X Y \longleftarrow \nabla_X Y$ dipende da $X(p)$ e da Y in $U(p)$
- $T(X,Y) \longleftarrow T(X,Y)(p)$ dipende solo da $X(p)$ e $Y(p)$

T è $C^\infty(M)$-lineare in entrambe le componenti:

$$T(fX, gY) = fg\,T(X,Y)$$

∇ è $C^\infty(M)$-lineare su X $\qquad \nabla_{fX} Y = f \nabla_X Y$

Teo: (M,g) pR $\exists!\ \nabla$ simmetrica e compatibile

In carte $$\Gamma_{ij}^{k} = g^{ke}\left(\frac{\partial g_{je}}{\partial x^i} + \frac{\partial g_{ie}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^e}\right) \star$$

dim:

Usando compatibilità e simmetria si vede che deve valere $\star$

È facile vedere che se definisco ∇ usando $\star$ allora ∇ viene compatibile e simmetrica

$$\nabla g = 0$$

$$\Leftrightarrow \frac{\partial g}{\partial x} = g\Gamma + g\Gamma$$

Proprietà della connessione di Levi-Civita

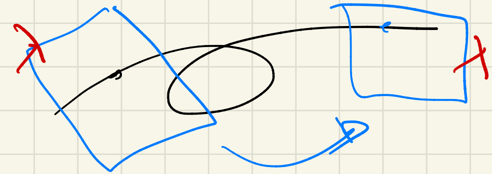
Naturalezza: Se $(M,g) \xrightarrow[\varphi]{\sim} (N,h)$ isometria

allora $$\varphi_*: \nabla^g \longrightarrow \nabla^h$$

Riscalamento: $g \dashrightarrow \lambda g \quad \lambda \neq 0$ costante

$$\nabla^{g} = \nabla^{\lambda g}$$

resta compatibile



$\Gamma(\gamma)_{t_0}^{t}$ è isom.

<u>Esempi</u>: $\mathbb{R}^n$ $\quad \mathbb{R}^{p,q}$ $\quad g = \begin{pmatrix} -I_q & \\ & I_p \end{pmatrix}$

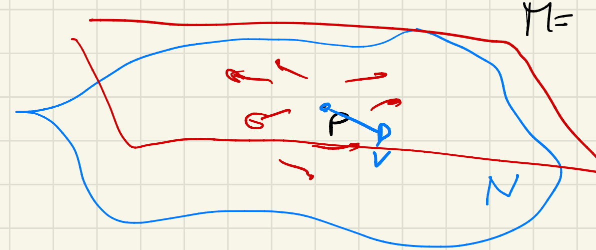
$$\Gamma_{ij}^{k} = 0 \implies \nabla_v^{g} X = \frac{\partial X}{\partial v} + v^i X^j \Gamma_{ij}^{k} e_k = \frac{\partial X}{\partial v}$$

$$\implies \nabla = \text{derivata direzionale}$$

Se $N \subseteq M$ sottovarietà pR di (M, g) pR $\quad \nabla^M$ su M $\quad \nabla^N$ su N

<u>Prop</u>: X campo in aperto di N

Lo estendo a M in modo arbitrario



Allora $\quad \nabla_v^N X = \pi(\nabla_v^M X)$

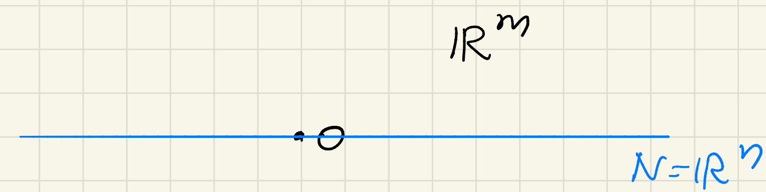
$\forall v \in T_pN \quad \underset{T_pN}{\cap} \quad \underset{T_pM}{\cap}$

$\forall p \in N$

$\pi : T_pM \to T_pN$ proiez. ortog.

dim: $\quad g\big|_{T_pM}$ non deg. $\Rightarrow T_pM = T_pN \oplus T_pN^{\perp}$

In carte $\quad M=\mathbb{R}^m \quad N=\mathbb{R}^n$



$p=0 \quad \mathbb{R}^n \subseteq \mathbb{R}^m$

$$g(0)=\left(\begin{array}{c|c} g^1 & 0 \\ \hline 0 & g^2 \end{array}\right)$$

Se $1\le i,j,k\le n \quad \Gamma_{ij}^k$ di N in $p=0$

$= \Gamma_{ij}^k$ di M in $p=0$

$$\Gamma_{ij}^k = g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right)$$

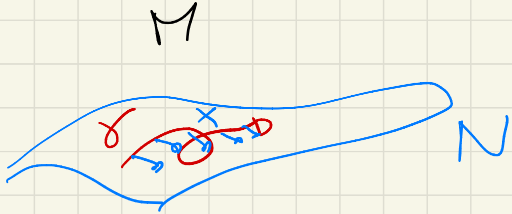
$v\in T_pN \quad \nabla^N_v X = \dfrac{\partial X}{\partial v} + v^i X^j \Gamma_{ij}^k e_k$

$\nabla^M_v X = \dfrac{\partial X}{\partial v} + v^i X^j \Gamma_{ij}^k e_k$

$\Rightarrow \nabla^N_v X = \pi\left(\nabla^M_v X\right)$

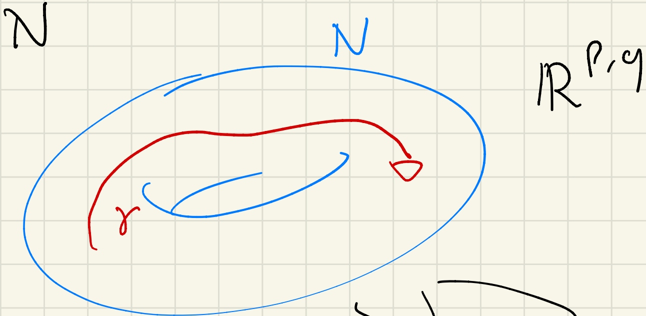
$\square$

Cor:



$$D^N_t X = \pi\left(D^M_t X\right)$$

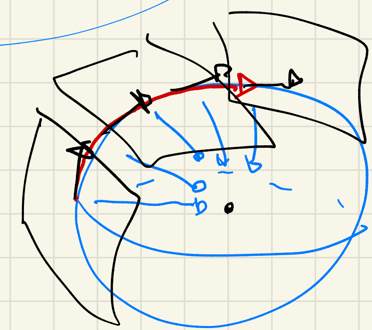
Cor: $N \subseteq \mathbb{R}^{p,q}$ $\qquad \gamma : I \to N$



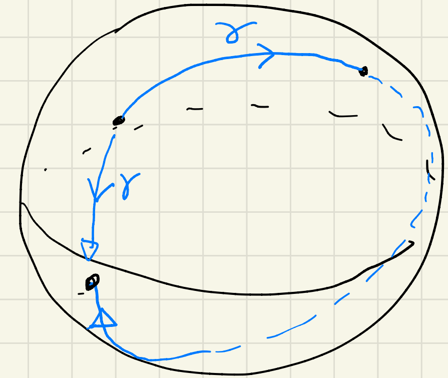
$X : I \to TN$ campo su γ

è parallelo $\iff X'(t) \perp T_{\gamma(t)} N$

derivata di X in $\mathbb{R}^{p,q}$



GEODETICHE



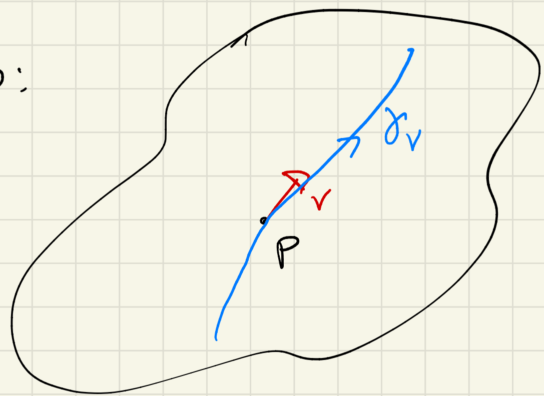
Def: (M, ∇)

Una curva $\gamma : I \to M$ è GEODETICA se $\gamma'(t)$ è parallelo.

Es: $\gamma : I \to M$ costante è geodetica (cioè $\dot{\gamma} = 0$)

Prop:

M



(M, ∇) $\forall p \in M$, $\forall v \in T_pM$

$\exists!$ γ_v geodetica massimale

con $\gamma_v(0) = p$, $\gamma_v'(0) = v$

$\gamma_v : I_v \to M$ $\quad 0 \in I_v$

dim: in carte

$$\nabla_v X = \frac{\partial X}{\partial v} + v^i X^j \Gamma_{ij}^k e_k$$

γ

$$D_t X = \frac{dX}{dt} + \dot{\gamma}(t)^i X^j \Gamma_{ij}^k e_k$$

$$\gamma(t) = x(t)$$

$$\gamma'(t) = \dot{x}(t)$$

$$D_t X = \frac{dX}{dt} + \dot{x}^i X^j \Gamma_{ij}^k e_k$$

$$X = \dot{x}(t)$$

$$D_t \dot{x} = \ddot{x} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k e_k = 0$$

$$\ddot{x} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k e_k = 0$$

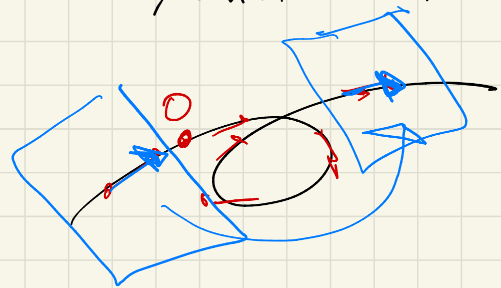
$$\ddot{x}_k + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0 \qquad \forall k$$

$$\ddot{x}_k = -\dot{x}^i \dot{x}^j \Gamma_{ij}^k$$

$\exists!$ soluzione con dati iniziali: $x(0) = p \qquad \dot{x}(0) = v$

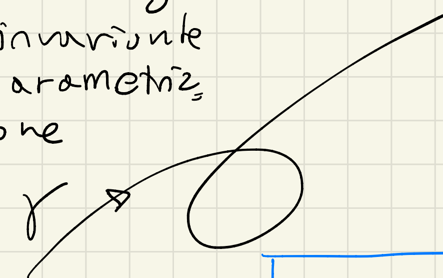
Oss: Se γ geod è non banale, allora è immersione $\left(\dot\gamma(t) \neq 0 \ \forall t\right)$

dim: Se $\dot\gamma(t) = 0$ per un $t \in I$ allora $\dot\gamma(t) = 0 \ \forall t \in I$.



Oss: $\|\dot\gamma(t)\|$ è cost.

Oss: Essere geod. non è invariante per riparametrizzazione



Prop: Se γ è geodetica, allora $\forall c \neq 0$ $\forall d \in \mathbb{R}$ $\eta(t) = \gamma(ct + d)$ è geodetica

dim: X campo parallelo su γ $\longrightarrow$ cX è parallelo

NON BASTA IN REALTÀ

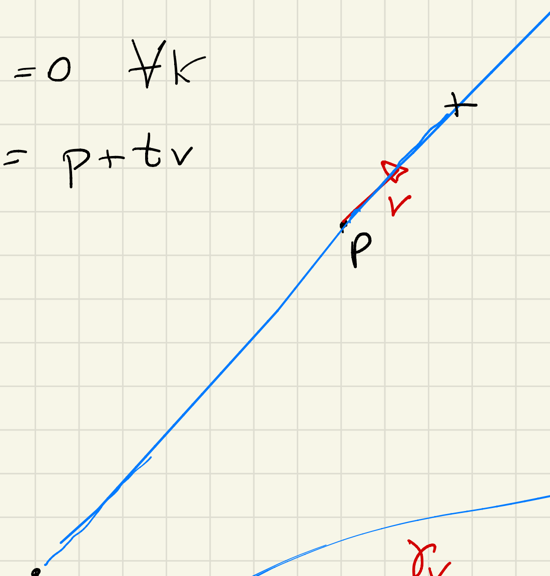
$\gamma_v(ct) = \gamma_{cv}(t)$
p.l.a. $\|v\| = 1$
si può fare se $\|v\| \neq 0$

$$c^2 \nabla_v X = \nabla_{cv} cX \qquad \nabla_v X = 0 \Rightarrow \nabla_{cv} cX = 0$$

<u>Esempi</u>: $\mathbb{R}^{p,q}$ $\quad \Gamma^k_{ij} = 0 \quad \ddot{x}_k = 0 \quad \forall k$

Le geodetiche sono $x(t) = p + tv$

Sono definite su $\mathbb{R}$



$N \subseteq \mathbb{R}^{p,q}$ sottovarietà pR

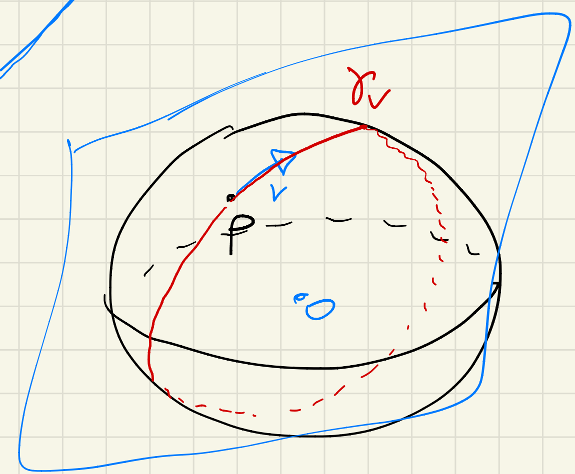
$\nabla_v = \frac{\partial}{\partial v}$

$x : I \to N$ geodetica

$\iff \quad \ddot{x}(t) \perp T_{x(t)} N$

<u>SFERA</u>: $S^n \subseteq \mathbb{R}^{n+1}$ $\quad v$ unitario

$$\gamma_v(t) = \cos t \cdot p + \operatorname{sen} t \cdot v$$



1) è geod: $\ddot{\gamma}_v(t) \perp T_{\gamma(t)} = \gamma(t)^{\perp}$

infatti $\ddot{\gamma}_v(t) = -\gamma_v(t)$

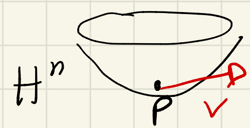
2) $\gamma_v(0) = p \qquad \dot{\gamma}_v(0) = v$

0) $\gamma_v(t) \in S^n$

<u>SPAZIO IPERBOLICO</u>

$\|v\|=1$

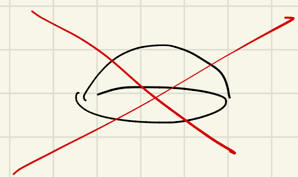
$H^n \subseteq \mathbb{R}^{n,1} \qquad \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & \ddots & \\ & & & & +1 \end{pmatrix}$



$$\gamma_v(t) = \cosh t \cdot p + \operatorname{senh} t \cdot v$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

$$\operatorname{senh} t = \frac{e^t - e^{-t}}{2}$$



Una geodetica è di tipo

TEMPO, SPAZIO o LUCE

a seconda di $\gamma'(t)$.

(non dip. da t)

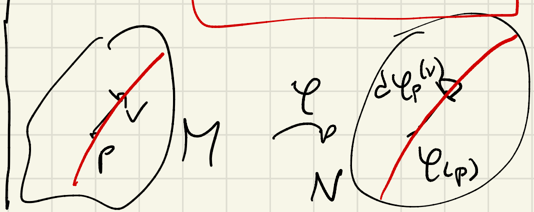
Se $\varphi : (M, g) \xrightarrow{\sim} (N, h)$

isometria

manda geodetiche in geodetiche

γ in M geod $\Rightarrow \varphi \circ \gamma$ geod in N

$$\boxed{\varphi \circ \gamma_v = \gamma_{d\varphi_p(v)}}$$



$\cosh^2 t - \operatorname{senh}^2 t = 1$ $\quad d \operatorname{senh} = \cosh$ $\quad d \cosh = \operatorname{senh}$

1) $\bar{e}$ geod. $\quad \ddot{\gamma}_v(t) = \gamma_v(t)$ $\quad T_{\gamma_v(t)} H^n = \gamma_{v(t)}^{\perp}$

2) $\gamma_v(0) = p$ $\quad \gamma_v(1) = v$ $\quad$ or

0) $\gamma_v(t) \in H^n$ $\quad$ (es)